On Hyperbolic Profinite Groups

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November 22, 2012

Abstract

In this note, we explore the notion of hyperbolicity of finitely and infinitely generated groups in case of profinite groups. Some applications to Diophantine geometry are suggested. In particular, we reformulate certain problems in Diophantine geometry in term of hyperbolic profinite groups.

Introduction

Gromov's theory of hyperbolic groups studies finitely generated groups whose Cayley graphs are hyperbolic metric spaces when equipped with word metric. This notion is independent of choices made for finitely many generators. We intend to extend this theory to infinitely generated groups and also to profinite groups. The purpose of this paper is to imply that hyperbolicy of profinite groups has applications in finiteness results in Diophantine geometry.

The notion of etale fundamental group introduced by Grothendieck, links the notion of Galois group with its function field counterpart which is geometric fundamental group [Gt]. By "Grothendieck's anabelian conjecture" the arithmetic fundamental group of a hyperbolic algebraic curve completely determines the algebraic structure of the curve. More precisely, if X is an algebraic variety over a field k we have the following exact sequence

$$0 \longrightarrow \pi_1(X_{\bar{k}}) \longrightarrow \pi_1(X_k) \longrightarrow Gal(\bar{k}/k) \longrightarrow 0$$

where $\pi_1(X_{\bar{k}})$ is the geometric fundamental group and $\pi_1(X_k)$ is the arithmetic fundamental group. This determines a homomorphism

$$\pi_1(X_k) \to Aut(\pi_1(X_{\bar{k}}))$$

by conjugating the group $\pi_1(X_{\bar{k}})$ by elements of $\pi_1(X_k)$ and then, after taking quotients, one obtains a homomorphism landing in the outer automorphism group

$$\rho_X : Gal(\bar{k}/k) \to Out(\pi_1(X_{\bar{k}}))$$

To consider the outer action of Galois group is equivalent to considering $\pi_1(X_k)$ as an extension of $Gal(\bar{k}/k)$.

By Grothendieck's "fundamental conjecture", if K is finitely generated over the prime field and if X is a hyperbolic algebraic variety, or in other words sufficiently non-abelian, ρ_X determines X over k. Grothendieck's "hom conjecture" states that if X and Y are algebraic varieties over a finitely generated field K, then the natural map between dominant morphisms from X to Y over K to Galois compatible open homomorphisms between the arithmetic fundamental groups

$$Hom(X,Y) \to Hom_{Gal(\bar{k}/k)}(\pi_1(X_k), \pi_1(Y_k))$$

induces a bijection if right hand side is considered up to composition with an inner automorphism of $\pi_1(Y_k)$. According to "section conjecture", splittings of the above short exact sequence are in one to one correspondence with the rational points of the projectivized variety. Moreover, sections corresponding to points at infinity are group theoretically characterized. The are several more explicit versions of this general philosophy, similar to "fundamental conjecture", "hom conjecture" and the "section conjecture". We refer the reader to [Na-Ta-Mo] for an exposition.

The known half of the section conjecture, tells us that to any point of X defined over k one can associate a splitting as above and this association is injective. Therefore, in order to prove finiteness of the rational points it is enough to show that $\pi_1(X_k)$ has only finitely many splittings as an extension of $Gal(\bar{k}/k)$ by $\pi_1(X_{\bar{k}})$. This brings us to the realm of profinite group theory. We suggest that hyperbolicity of $\pi_1(X_{\bar{k}})$ is enough to get such finiteness results. This leads us to the following conjecture which implies Mordell's conjecture proved by Falthings:

Conjecture 0.1 *Let H denote a finitely generated profinite hyperbolic group, and let G denote a split extension of the full Galois group over a number field k by the hyperbolic group H*

$$0 \to H \to G \to Gal(\bar{k}/k) \to 0$$

Then there are only finitely many H-conjugacy classes of splittings of the above short exact sequence.

Organization of the paper is as follows. In the first section, we review basic definition of hyperbolic metric spaces and hyperbolic groups. In the second section, we try to extend the notion of hyperbolicity to infinitely generated groups. We review needed background in profinite groups and pro-p groups as a special class of profinite groups in the forth section. In the same section, we define finitely generated hyperbolic profinite groups, and try to extend this notion to infinitely generated groups. The final section contains a profinite hyperbolic group theory formulation of Diophantine geometry.

1 Preliminaries on hyperbolic groups

This paper contains several conjectures which are not to be used in the arguments coming afterwards. They are presented here to indicate our suggested framework in which one may consider the theory of hyperbolic profinite groups.

Motivated by hyperbolic phenomena in the geometry of manifolds and spaces of negative curvature, Gromov defined the notion of hyperbolic finitely generated groups. Trying to make the formulation of this concept independent of curvature, he invented the notion of a quasi-isometry between metric spaces [Gm].

Definition 1.1 *Let (X, d) be a metric space. The Gromov product of points x and y of X with respect to a base point x_0 is defined to be the real number*

$$(x.y) = (x.y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$$

The metric space X is called 'Gromov hyperbolic' or simply 'hyperbolic', if it is δ-hyperbolic for some $\delta \geq 0$ which means that

$$(x.y) \geq min((x.z), (y.z)) - \delta$$

should hold for all $x, y, z \in X$ and every choice of basepoint x_0. A geodesic metric space X is hyperbolic if and only if for some $\delta \geq 0$ every geodesic triangle is δ-thin, meaning that every side is contained in the δ-neighborhood of the union of two other sides.

Important examples of hyperbolic spaces are: the hyperbolic upper half-space $\mathbb{H}^n$ and all simply-connected Riemannian manifolds with sectional curvature bounded above but away from zero, $\mathbb{R}$-trees and Cayley graphs of hyperbolic groups.

Definition 1.2 *A finitely generated group G is called hyperbolic, if for some finite generating set Γ the Cayley graph of G with respect to Γ is a hyperbolic space with word metric.*

In order to show that hyperbolicity is well-defined one has to use the notion of quasi-isometries between metric spaces and the fact that hyperbolicity is a quasi-isometric invariant. Important examples of hyperbolic groups are finite groups, free groups of finite rank, fundamental groups of compact Riemannian manifolds with negative sectional curvature, and groups properly discontinuously acting on hyperbolic spaces with compact quotient. Here are some properties hyperbolic groups: A hyperbolic group is finitely presented. It can not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A finite index subgroup is hyperbolic if and only if the ambient group is. A hyperbolic groups acts freely and transitively on a contractible finite dimensional locally finite polyhedral complex. As a consequence, it contains only finitely many conjugacy classes of torsion elements.

Definition 1.3 *Let G be an infinitely generated discrete group, which is an increasing union of a chain of its finitely generated subgroups G_i with $G_i \subset G_j$ for $i < j$. For Γ_i a finite generating set for G_i such that $\Gamma_i \subset \Gamma_j$ for $i < j$, the Cayley graph of G with respect to $\cup_i \Gamma_i$ is defined to be the increasing union of Cayley graphs of G_i with respect to Γ_i. The group G is called hyperbolic, if it is a union of finitely generated subgroups G_i as above, such that its Cayley graph with respect to $\cup_i \Gamma_i$ equipped with word metric is a hyperbolic metric space. Equivalently, the Cayley graphs of G_i with respect to Γ_i are δ-hyperbolic for a non-negative δ independent of i.*

This way, infinitely generated discrete groups could inherit hyperbolicity structures from finitely generated groups. Our leading example for infinitely generated hyperbolic groups is infinitely generated free group. Hyperbolicity of an infinitely generated discrete group shall be proved to be independent of the choices made for a chain of finitely generated hyperbolic subgroups and their generators.

2 Preliminaries on profinite groups

Interest in profinite groups originates in the study of the Galois groups of infinite extensions of fields. Galois groups naturally carry 'Krull' topology under which they become Hausdorff compact totally disconnected topological groups. These properties characterize profinite groups and indeed, profinite groups are precisely Galois groups.

Definition 2.1 *Let C denote a class of finite groups. A pro-C group G is an inverse limit of a surjective inverse system of groups G_i in C with discrete topology.*

$$G = \lim_{\leftarrow} G_i$$

We say that G is topologically generated by a finite subset Γ if its image in G_i generates G_i, or equivalently if the closure of the group generated by Γ is G. In this case, we say that G is the pro-C completion of the group generated by Γ.

In case C is the class of all finite groups, then G is called a profinite group and if C is the class of p-groups, then G is called a pro-p group. Since we intend to discuss hyperbolic profinite groups, we shall first recognize the class of free pro-C groups as the most important class of hyperbolic profinite groups.

Definition 2.2 *For any profinite space X there exists a unique pro-C group $F_C(X)$ containing X as a profinite subset which satisfies the following universal property: Any continuous map $X \to G$ to a pro-C group G whose image generates G, uniquely breaks through inclusion $X \hookrightarrow F$ and a continuous homomorphism $F \to G$. $F_C(X)$ is called the free pro-C group on the profinite basis X.*

In the case of finitely generated free pro-C groups, any generating set with the same cardinality as the basis is again a basis. In fact, pro-C completion of the free group generated by finite basis X is the free pro-C group on the basis X. Free pro-C groups are examples of free pro-C products.

Definition 2.3 *A family of homomorphisms $\rho_i : H_i \to H$ between pro-C groups, for $i \in I$, is called convergent to 1 if $\rho_i(H_i)$ is convergent to 1 which means that any open neighborhood of 1 contains almost all $\rho_i(H_i)$. The free pro-C product of a family of H_i is a pro-C group $H = *_{i\in I} H_i$ together with a convergent family of homomorphisms $\rho_i : H_i \to H$ which is universal among all such families targeting in any group H'. Free pro-C products are unique up to isomorphisms.*

Subgroups of finite index in free pro-C products are again free pro-C products, which could be an indication of the fact that free products being similar to free groups are usually hyperbolic groups. For example, free products of finitely many finitely generated hyperbolic groups are again hyperbolic groups.

3 Hyperbolic profinite groups

Now, we shall try to introduce a profinite version of the hyperbolic phenomena. In order to discuss hyperbolicity of a profinite group, we need a geometric object to replace the role of Cayley graph of a hyperbolic group.

Definition 3.1 *Let G be a residually finite profinite group, which is isomorphic to inverse limit of all its finite images G_i of G, and let Γ denote a finite generating set. The Cayley graph G_Γ of G with respect to Γ is defined to be the Cayley graph of the subgroup generated by Γ together with all its translates in G. One can think of G_Γ as the boundry of the union of Cayley graphs of G_i with respect to Γ_i, where Γ_i denotes the image of Γ in G_i.*

Given a residually finite hyperbolic group H whose Cayley graph with respect to a finite generating set Γ is δ-hyperbolic for some positive δ, all translates of this Cayley graph in the profinite completion are also δ-hyperbolic. This motivates the following definition:

Definition 3.2 *A finitely generated profinite group G is hyperbolic, if its Cayley graph with respect to a finite topological generating set equipped with word metric is a hyperbolic metric space.*

Proposition 3.3 *Free pro-C products of finitely many finitely hyperbolic pro-C groups G_i are again hyperbolic pro-C groups.*

Proof. Let Γ_i denote a finite generating set for G_i for $i \in I$. If the Cayley graphs of G_i with respect to Γ_i are δ-hyperbolic, then the Cayley graph of $*_{i\in I}G_i$ with respect to $\cup_i\Gamma_i$ will also be δ-hyperbolic. $\square$

Definition 3.4 *Let G be an infinitely generated profinite group, which is an increasing union of a chain of its finitely generated profinite subgroups G_i with $G_i \subset G_j$ for $i < j$. For Γ_i a finite generating set for G_i such that $\Gamma_i \subset \Gamma_j$ for $i < j$, the Cayley graph of G with respect to $\cup_i\Gamma_i$ is defined to be the increasing union of Cayley graphs of G_i with respect to Γ_i. The group G as above is called hyperbolic, if it is a union of finitely generated profinite groups G_i as above, such that its Cayley graph with respect to $\cup_i G_i$ equipped with word metric is a hyperbolic metric space.*

Hyperbolicity of a finitely generated and an infinitely generated profinite group shall be proved to be independent of the choices made.

4 Profinite hyperbolicity and Diophantine geometry

We shall start with the following basic conjecture:

Conjecture 4.1 *For any number field k the full Galois group $Gal(\bar{k}/k)$ is a hyperbolic profinite group.*

For an arbitrary curve X defined over the base field k, the algebraic fundamental group $\pi_1^{alg}(X_k)$ is defined by Grothendieck as

$$\lim Gal(K'/K)$$

where K is the function field of X and K' runs over all Galois extensions of K such that the corresponding curve X' is etale over X. For example, $\pi_1^{alg}(\mathbb{P}^1) = 1$ and for every elliptic curve E we have

$$\pi_1^{alg}(E) = \prod_{\ell} \mathbb{Z}_\ell \times \mathbb{Z}_\ell.$$

Grothendieck proved that, for a curve X of genus g defined over an algebraically closed base field $\bar{k}$, the algebraic fundamental group of X is isomorphic to the completion of the ordinary topological fundamental group of $\bar{X}$ over $\mathbb{C}$ [Gt]. One can also define the ℓ-adic algebraic fundamental group $\pi_{1,\ell}^{alg}(X_k)$ as the limit

$$\lim Gal(K'/K)$$

where K is the function field of X and K' runs over all Galois extensions of K of degree ℓ^n such that the corresponding curve X' is etale over X. Grothendieck also associated to X the following exact sequence

$$0 \longrightarrow \pi_1(X_{\bar{k}}) \longrightarrow \pi_1(X_k) \longrightarrow Gal(\bar{k}/k) \longrightarrow 0$$

which indicates that the arithmetic fundamental group is an extension of the Galois group by the geometric fundamental group of X. To a rational point on X, one associates a $\pi_1(X_{\bar{k}})$-conjugacy class of maps $Gal(\bar{k}/k) \to \pi_1(X_k)$ splitting the exact sequence. It is expected that general hyperbolicity assumptions on X imply finiteness of rational points.

There exists a pro-p version of the Grothendieck short exact sequence for a curve defined over any finitely generated field

$$0 \longrightarrow \pi_1^{(p)}(X_{\bar{k}}) \longrightarrow \pi_1^{(p)}(X_k) \longrightarrow Gal(\bar{k}/k)(p) \longrightarrow 0$$

and the philosophy of Grothendieck's section conjecture could be also implied in this case. In the previous section, many examples are introduced where we get hyperbolicity of certain pro-p Galois groups for p outside a finite set of primes S. Together with hyperbolicity of the corresponding $\pi_1^{(p)}(X_{\bar{k}})$ one is lead to the following conjecture

Conjecture 4.2 *Let k denote a finitely generated field and X a hyperbolic algebraic curve defined over k, and assume that $Gal(\bar{k}/k)(p)$ is hyperbolic for all p outside a finite set of primes S, then the number of points of X defined over A_S is finite, where A_S denotes the integral closure of $\mathbb{Z}$ in the maximal extension of k unramified outside S.*

We also believe the following relation between pro-p hyperbolicity and profinite hyperbolicity must be true:

Conjecture 4.3 *Let H denote a finitely generated profinite group. If $H(p)$ is hyperbolic profinite for all primes p, then H is also a hyperbolic profinite group.*

This would not be a reasonable conjecture until one proves hyperbolicity of H implies hyperbolicity of H_p. If such a result be true we get Siegel's Theorem out of previous conjectures.

In a more abstract setting, one can also think of a profinite group Γ surjecting to the Galois group, as an "arithmetic" space

$$\Gamma \longrightarrow Gal(\bar{k}/k) \longrightarrow 0$$

and ask for pure algebraic hyperbolicity conditions implying the finiteness of splittings $Gal(\bar{K}/K) \to \Gamma$ of the corresponding short exact sequence. This would be a generalization of the Mordell conjecture. Using such algebraic methods one may be able to prove the following folklore more strong version of Mordell's conjecture:

Conjecture 4.4 *The number of rational points of any hyperbolic curve defined over a number field, is bounded above by a bound depending only on the genus and the number field.*

This result is proved in the function field case using Model theory. Up to know there exists no number theoretical proof of the Mordell's conjecture in the function field case.

References

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